ON THE IMPOSSIBILITY OF STABILIZING A SYSTEM IN THE MEAN-SQUARE BY RANDOM PERTURBATION OF ITS PARAMETERS

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At the present time there are sufficiently many works in which questions of the stability of systems with parameters which are random functions of an independent variable are considered (such as [1 to 13], for example). Stability in the mean-square is examined in many of them.

The following question was also examined in [5 and 6]. Let some solution of a second order differential equation with constant coefficients increase without limit together with the argument. Is it possible to achieve boundedness of a solution with the same initial conditions in the mean-square if a random function $\alpha(t)$ is appended to one of the coefficients of this equation? The correct answer, obtained in [6], in which the function $\alpha(t)$ satisfies the same demands as in [5] ($\alpha(t)$ describes Gaussian white noise, say), turns out to be negative.

In [10], in particular, the system of equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} (c_{ij} + r_{ij}(t))\, x_j(t)$$

where the c_{ij} are constants and the $r_{ij}(t)$ random functions, was analyzed. It turned out that this system is not stable in the mean if the functions $r_{ij}(t)$ describe Gaussian white noise and the corresponding deterministic system is unstable.

The following problem is posed herein.

1. Let some deterministic system be described by Equation

$$u^{(n)}(t) + a_{n-1}u^{(n-1)}(t) + \ldots + a_0 u(t) = f(t) \qquad (n > 1,\ 0 \leqslant t \leqslant \infty) \qquad (1.1)$$

in which the coefficients $a_0, a_1, \ldots, a_{n-1}$ are real constants, the function $f(t)$ is real and

$$|f(t)| < \frac{N}{(1+\varkappa t)^{\rho}} \qquad (\rho > 0,\ \varkappa \geqslant 0) \qquad (1.2)$$

The quantities N, $\varkappa$ and ρ are independent of t.

Replacement of some coefficient a_j $(j=0, 1, \ldots, n-2)$ by $a_j - \alpha_j(t)$, where $\alpha_j(t)$ is a random function, leads to Equation

$$u^{(n)}(t, \alpha_j) + a_{n-1}u^{(n-1)}(t, \alpha_j) + \ldots + a_0 u(t, \alpha_j) = f(t) + \alpha_j(t)\, u^{(j)}(t, \alpha_j) \quad (j = 0, 1, \ldots, n-2) \tag{1.3}$$

The following is assumed relative to the random function $\alpha_j(t)$.

1) The autocorrelation range a of the random process $\alpha_j(t)$ is zero (*)

2) $\langle \alpha_j(t)\rangle = 0, \quad \langle \alpha_j(t)\,\alpha_j(t+\tau)\rangle = S_j\delta(\tau) \quad (-\infty \leqslant t, \tau \leqslant \infty; S_j = \text{const} > 0)$

Here and henceforth, the angle brackets denote the average over the ensemble. Conditions (1) and (2) are an idealization of the following conditions (as $a \to 0$).

1a) The random process $\alpha_j(t)$ has an autocorrelation range $a > 0$;

2a) $\langle \alpha_j(t)\rangle = 0, \quad \int_{-a}^{a} \langle \alpha_j(t)\,\alpha_j(t+\tau)\rangle\, d\tau = S_j, \quad \mathrm{P}\left[\,|\alpha_j(t)| < \frac{\gamma_j}{\sqrt{a}}\right] = 1$

where $\gamma_j = \text{const} > 0$, P is probability; $-\infty \leqslant t \leqslant \infty$.

It is necessary to clarify the relation between the asymptotes of any particular solution $u(t)$ of (1.1) and the mean square $\langle u^2(t, \alpha_j)\rangle$ of the solution $u(t; \alpha_j)$ of (1.2), obtained under the same initial conditions as for $u(t)$.

The obtained results reduce to the following two theorems.

Theorem 1.1. If the real part of at least one characteristic number of (1.1) is positive, then the mean square $\langle u^2(t, \alpha_j)\rangle$ of any particular solution of (1.3) (with the random function $\alpha_j(t)$ possessing properties (1) and (2)) increases without limit with t for any values S_j.

Theorem 1.2. If the characteristic numbers λ_i of (1.1) are such that $\operatorname{Re}\lambda_i < 0$ $(i = 1, \ldots, n)$, then the mean square $\langle u^2(t, \alpha_j)\rangle$ of any particular solution of (1.3) (with the random function $\alpha_j(t)$ possessing properties (1) and (2)) will increase unboundedly with t if and only if

$$S_j > S_j^* = \left[\frac{(-1)^j}{2\pi i}\int_{-i\infty}^{i\infty} \frac{z^{2j}\,dz}{L_n(z)\,L_n(-z)}\right]^{-1} > 0$$

Here $L_n(\lambda)$ is the characteristic polynomial of (1.1) (**)

In case $S_j < S_j^*$ and $\varkappa > 0$

$$\lim \langle u^2(t, \alpha_j)\rangle = 0 \quad \text{for } t \to \infty$$

To prove these Theorems let us derive an expression for $\langle u^2(t, \alpha_j)\rangle$.

2. In (1.3) let the random function $\alpha_j(t)$ satisfy the conditions (1a) and (2a) from Section 1. Equation (1.3) is equivalent to the relation

$$u(t, \alpha_j) = u(t) + \int_0^t W(t-q)\,\alpha_j(q)\,u^{(j)}(q, \alpha_j)\,dq, \qquad W(t-q) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{p(t-q)}}{L_n(p)}\,dp \tag{2.1}$$

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*) A quantity $a > 0$ such that for any fixed t and $|\tau| > a$, the random variables $\alpha_j(t)$ and $\alpha_j(t+\tau)$ are independent is called the autocorrelation range.

**) For example, for the second order equation

$$u\ (t, \alpha_0) + 2\beta u\ (t, \alpha_0) + \omega_0^2\,[1 + \alpha_0(t)]\,u(t, \alpha_0) = 0 \ (\beta > 0)$$

the necessary and sufficient condition for boundedness of any particular solution in the mean is $S_0 < 4\beta/\omega_0^2$, which agrees completely with [6].

Here $u(t)$ is any particular solution (*) of (1.1) and $W(t-q)$ is a Cauchy function; $\gamma > \operatorname{Re}\Lambda$, where Λ is the zero of the characteristic polynomial $L_n(\lambda)$ of (1.1) which has the greatest real part (or one of such zeroes). As is known, the function $W(t)e^{-\gamma t} \to 0$ as $t \to \infty$ no matter what the $\gamma > \operatorname{Re}\Lambda$.

The integral equation

$$u^{(j)}(t, \alpha_j) = u^{(j)}(t) + \int_0^t W^{(j)}(t-q)\,\alpha_j(q)\,u^{(j)}(q, \alpha_j)\,dq \qquad (j = 0, 1, \ldots, n-2) \qquad (2.2)$$

is easily obtained from (2.1).

As is known, the solution of this equation has the form

$$u^{(j)}(t, \alpha_j) = \sum_{s=0}^{\infty} U_s(t, \alpha_j), \qquad U_0(t, \alpha_j) = u^{(j)}(t)$$

$$U^s(t, \alpha_j) = \int_{\Gamma_s} W_{js}(y)\,\alpha_{js}(y)\,u^{(j)}(y_s)\,dy \qquad (s = 1, 2, \ldots$$

Here $y = \{y_1, \ldots, y_s\}$ is an s-dimensional vector; Γ_s a domain in s-dimensional space divided up by means of the inequalities

$$t \geqslant y_1 \geqslant y_2 \geqslant \ldots \geqslant y_s \geqslant 0;$$

$$\alpha_{js}(y) = \alpha_j(y_1)\,\alpha_j(y_2)\ldots\alpha_j(y_s), \qquad W_{js}(y) = W^{(j)}(t-y_1)\,W^{(j)}(y_1-y_2)\ldots W^{(j)}(y_{s-1}-y_s)$$

Substitution of (2.2) for $u^{(j)}(t, \alpha_j)$ in (2.1) leads to the relation

$$u(t, \alpha_j) = \sum_{s=0}^{\infty} V_s(t, \alpha_j)$$

where

$$V_0(t, \alpha_j) = u(t), \qquad V_s(t, \alpha_j) = \int_{\Gamma_s} G_{js}(y)\,\alpha_{js}(y)\,u^{(j)}(y_s)\,dy \qquad (s = 1, 2, \ldots)$$

$$G_{j1}(y) = W(t-y_1)$$

$$G_{j1}(y) = W(t-y_1)\,W^{(j)}(y_1-y_2)\,W^{(j)}(y_2-y_3)\ldots W^{(j)}(y_{s-1}-y_s) \qquad (s = 2, 3, \ldots)$$

Hence

$$\langle u^2(t, \alpha_j)\rangle = u^2(t) + \sum_{s,\,\sigma=0}^{\infty} \langle V_s(t, \alpha_j)\,V_\sigma(t, \alpha_j)\rangle \qquad (s + \sigma \geqslant 2). \qquad (2.3)$$

This expression is simplified substantially if the autocorrelation range a approaches zero, i.e. if we go from conditions (1a), (2a) of Section 1 to conditions (1) and (2).

The following must here be kept in mind.

a) Each term of the sum in (2.3) is some integral. Those terms for which $n = s + \sigma$ is odd vanish together with a. This follows from estimates, by the method expounded in [11], of the sum of the volumes of those parts of the domain of integration Γ_s where the integrand is not zero.

b) The contribution in any of the remaining terms, from integration over those portions of the domain Γ_s which do not correspond to grouping of the arguments in twos (**), vanishes together with a. This follows from

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*) For simplicity, the initial conditions for $u(t)$ are assumed to be deterministic.

**) Grouping of the arguments in twos is the grouping of $s + \sigma$ arguments $(y_1, y_2)(y_3, y_4)\ldots(y_{s+\sigma-1}, y_{s+\sigma})$, such that

$$y_{2k+1} - y_{2k+2} \leqslant a, \qquad y_{2k+2} - y_{2k+3} > a \qquad \left(k = 0, 1, \ldots, \frac{s+\sigma}{2} - 1\right)$$

estimates of the sum of the volumes of such portions of Γ_n, made by the same method.

c) The contribution from integration over the portions of Γ_n corresponding to such a grouping of the arguments in pairs, for which at least one pair of arguments coincides with the arguments of some of the functions

$$W^{(j)}(y_i - y_{i+1}) \ (j = 0, 1, \ldots, n-2),$$

also approaches zero. This latter is the result of an exact calculation of such a contribution when passing to the properties (1) and (2) as a limit, if it is here taken into account that $W^{(j)}(0) = 0$ for $j = 0, 1, \ldots, n-2$.

Thus only those terms of the sum in (2.3) for which $s = \sigma$ do not vanish together with a. In evaluating them it is necessary to integrate only over those portions of the domain Γ_n which correspond to grouping of the arguments in pairs, and moreover, such that none of the pairs coincides with the arguments of any of the functions $W^{(j)}$.

Then in the limit as $a \to 0$ the function $\langle u^2(t, \alpha_j)\rangle$ takes the form

$$\langle u^2(t, \alpha_j)\rangle = u^2(t) + \sum_{n=1}^{\infty} S_j^n \int_{\Gamma_n} [G_{jn}(y)\, u^{(j)}(y_n)]^2\, dy \tag{2.4}$$

It is seen already from this formula that if $u^2(t)$ increases without limit together with t, then $\langle u^2(t, \alpha_j)\rangle$ will also increase without limit.

It is easy to note that each term of the series in the last formula is a convolution, which permits the summation of this series by using the Laplace transform. Indeed

$$L\{\langle u^2(t, \alpha_j)\rangle\} = L\{u^2(t)\} + L\{W^2(t)\}\, L\{[u^{(j)}(t)]^2\} \sum_{n=1}^{\infty} S_j^n (L\{[W^{(j)}(t)]^2\})^{n-1}$$

where the symbol $L\{\ldots\}$ denotes the Laplace transform. Hence

$$\langle u^2(t, \alpha_j)\rangle = \tag{2.5}$$

$$= \frac{1}{2\pi i} \int_{p_0 - i\infty}^{p_0 + i\infty} \frac{L\{u^2(t)\} + S_j(L\{W^2(t)\}\, L\{[u^{(j)}(t)]^2\} - L\{[W^{(j)}(t)]^2\}\, L\{u^2(t)\})}{1 - S_j L\{[W^{(j)}(t)]^2\}}\, e^{pt} dp$$

Here $p_0 > \operatorname{Re} p_i$, where p_i are singularities of the integrand.

For $S_j = 0$ the right side of (2.5) becomes equal to $u^2(t)$ (it should be recalled that $u(t)$ is any particular solution of (1.1) and $u(t, \alpha_j)$ is the solution of (1.3) obtained under the same initial conditions as for $u(t)$.

3. Formula (2.5) permits the solution of the question of the boundedness of the function $\langle u^2(t, \alpha_j)\rangle$ by using elementary considerations.

It follows from (2.5) that if $S_j > 0$ and the characteristic numbers λ_i of (1.1) are such that $\operatorname{Re}\lambda_i < 0$ $(i = 1, \ldots, n)$, then the zeroes of the denominator are the only poles of the integrand in the half-plane $\operatorname{Re} p > 0$. Indeed, the functions

$$L\{[W^{(j)}(t)]^2\}, \quad L\{[u^{(j)}(t)]^2\} \ (j = 0, 1, \ldots, n-2)$$

then have no poles in the half-plane $\operatorname{Re} p > 0$: for $L\{[W^{(j)}(t)]^2\}$ this follows from the properties of the Cauchy function, for $L\{[u^{(j)}(t)]^2\}$, from the boundedness by the condition on the function $f(t)$ $(0 \leqslant t \leqslant \infty)$.

Hence, Equation

$$\Phi_j(p) \equiv L\{[W^{(j)}(t)]^2\} = S_j^{-1} \qquad (j = 0, 1, \ldots, n-2) \tag{3.1}$$

should be examined and the location of its roots on the p plane should be investigated as a function of the quantity $S_j > 0$ and the values of the characteristic numbers λ_i $(i = 1, \ldots, n)$ of (1.1).

It is easy to see that for any values of the numbers λ_i $(-\infty < \operatorname{Re}\lambda_i < \infty)$ the function $\Phi_j(p)$ has the following properties.

1. The function $\Phi_j(p)$ is defined and analytic in the half-plane $\operatorname{Re} p > 2 \operatorname{Re} \Lambda$, where Λ is the characteristic number of (1.1) which has the greatest real part.

2. The function $\Phi_j(p)$ is real on the real half-axis $p > 2 \operatorname{Re} \Lambda$.

3. The function $\Phi_j(p) \to 0$ as $p \to +\infty$.

4. If $p > 2 \operatorname{Re} \Lambda$, then $\lim \Phi_j(p) = +\infty$ as $p \to 2 \operatorname{Re} \Lambda$.

5. If $\operatorname{Re} \lambda_i < 0$ $(i = 1, \ldots, n)$, then the greatest absolute value of $\Phi_j(p)$ in the closed right half-plane is $\Phi_j(0) > 0$.

The last property follows from the fact that

$$\Phi_j(p) = \int_0^\infty [W^{(j)}(t)]^2 e^{-pt}\, dt$$

(since the function $W^{(j)}(t)$ is real, $0 \leqslant t \leqslant \infty$).

It follows from properties (2), (4), (1) and (3) that if $\operatorname{Re} \Lambda > 0$, then at least one root of (3.1) (*) lies on the half-axis $p > 2 \operatorname{Re} \Lambda$. The numerator of the integrand in (2.5) has the form

$$S_j L\{W^2(t)\} L\{[u^{(j)}(t)]^2\} e^{pt}$$

i.e. is greater than zero for real p, for the values of p which are zeroes of the denominator. This proves Theorem 1.1.

If $\operatorname{Re} \lambda_i < 0$ $(i = 1, \ldots, n)$, then at least one root of (3.1) is found in the half-plane $\operatorname{Re} p > 0$ if and only if

$$S_j > S_j^* = \Phi_j^{-1}(0) \tag{3.2}$$

In fact it follows from property (5) that for $S_j < S_j^*$ Equation (3.1) cannot have any roots in the closed right half-plane. If $S_j > S_j^*$, then from properties (5), (2), (1) and (3) of the function $\Phi_j(p)$ results the existence of at least one root of (3.1) on the half-axis $p > 0$

As is known, if

$$\operatorname{Re} \lambda_i < 0 \quad (i = 1, \ldots, n), \ |f(t)| < \frac{N}{(1 + \varkappa t)^\rho} \quad (\rho > 0, \varkappa \geqslant 0; 0 \leqslant t \leqslant \infty)$$

the following estimate is valid

$$|u^{(j)}(t)| < \frac{C_j}{(1 + \varkappa t)^\rho} \equiv M_j(t) \qquad (j = 0, 1, \ldots, n-2; 0 \leqslant t \leqslant \infty)$$

The quantities C_j are independent of t.

If we put $M_j^2(y_n)$ into (2.4) in place of $[u^{(j)}(y_n)]^2$, then the new series obtained as result of such a substitution converges to some function $F_j(t)$ such that $F_j(t) > \langle u^2(t, \alpha_j)\rangle$ $(0 \leqslant t \leqslant \infty)$. Each term of the series for $F_j(t)$ (just as the terms of the series for $\langle u^2(t, \alpha_j)\rangle$) is a convolution, which permits using the Laplace transformation to obtain the equality

$$L\{F_j(t)\} = \frac{L\{M_0^2(t)\} + S_j (L\{W^2(t)\} L\{M_j^2(t)\} - L\{[W^{(j)}(t)]^2\} L\{M_0^2(t)\})}{1 - S_j L\{[W^{(j)}(t)]^2\}}$$

$$(j = 0, 1, \ldots, n-2)$$

Applying the inversion formula to $L\{F_j(t)\}$ and investigating the deformations of the contour of integration admissible for $\varkappa = 0$ and $\varkappa > 0$, we may arrive at the following conclusion: If $\operatorname{Re} \lambda_i < 0$ $(i = 1, \ldots, n)$, the function $f(t)$ satisfies condition (1.2) and $S_j < S_j^*$ $(j = 0, 1, \ldots, n-2)$, then

$$\sup_t \langle u^2(t, \alpha_j)\rangle < \infty \quad (\varkappa \geqslant 0, 0 \leqslant t \leqslant \infty), \quad \lim_t \langle u^2(t, \alpha_j)\rangle - 0 \quad (\varkappa > 0, t \to \infty)$$

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*) Evidently this same deduction is valid also for $\operatorname{Re} \Lambda = 0$ if j is less than the multiplicity of the characteristic number Λ or $\operatorname{Im} \Lambda \neq 0$.

Theorem 1.2 is proved.

Let us evaluate the quantity $\Phi_j(0)$ which appears in (3.2). According to the definition of the functions $\Phi_j(p)$ and $W(t)$ (Formulas (3.1) and (2.1))

$$\Phi_j(0) = \int_0^\infty [W^{(j)}(t)]^2\,dt, \qquad W^{(j)}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{z^j e^{zt}dz}{L_n(z)}$$

$$(j = 0, 1, \ldots, n-2;\ \operatorname{Re}\Lambda < \gamma < 0)$$

After having substituted the expression for $W^{(j)}(t)$ in the first formula and having integrated with respect to t, the quantity $\Phi_j(0)$ is expressed by Formula

$$\Phi_j(0) = \frac{1}{(2\pi)^2}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} dz_1 \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} dz_2\, \frac{z_1 z_2}{L_n(z_1)\,L_n(z_2)\,(z_1+z_2)} \qquad \left(\begin{array}{c} j = 0, 1, \ldots, n-2 \\ \operatorname{Re}\Lambda < \gamma_1, \gamma_2 < 0 \end{array}\right)$$

Having evaluated the inner integral and permitted γ_1 to approach zero, we finally obtain

$$\Phi_j(0) = \frac{(-1)^j}{2\pi i}\int_{-i\infty}^{i\infty} \frac{z^{2j}\,dz}{L_n(z)\,L_n(-z)}$$

It is easy to see that the integrand takes on only real values on the contour of integration.

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